# Confidence Intervals

CIVL 7012/8012

# Sampling Distributions

- Because we typically can not evaluate an entire population to determine its parameters, we rely on estimators based on samples
- The estimators themselves make up a sampling distribution, and the variance of this distribution gives us information regarding the reliability of our estimate

$$E(\overline{X}) = \sum_{i=1}^{n}\left(\frac{1}{n}\right)\mu = \mu \quad \text{and} \quad \operatorname{var}(\overline{X}) = \sum_{i=1}^{n}\left(\frac{1}{n}\right)^2\sigma^2 = \frac{\sigma^2}{n}.$$

# The Central Limit Theorem

Suppose we have a population described by a random variable *X* with a mean $\mu$ and a standard deviation $\sigma$. We place no restrictions on the probability distribution of *X*. It may be normally distributed, uniformly distributed, exponentially distributed, it doesn’t matter.

Suppose we now take random samples from this population, each with a fixed and large sample size *n*. Each sample will have a sample mean $\bar{X}$, and this $\bar{X}$ will not, in general, be equal to the population mean $\mu$.

After repeated samplings, we will have built a population of $\bar{X}$s. The $\bar{X}$s are themselves random variables and they have their own probability distribution!

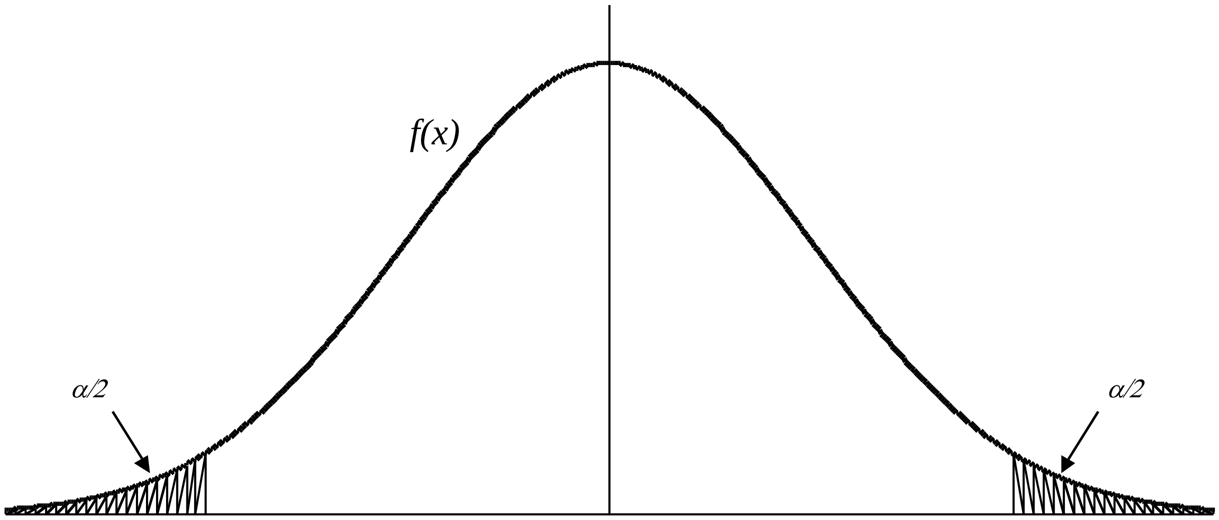
The ***Central Limit Theorem*** says that, as long as *n* is reasonably large,

$$\bar{X} : N\left[\mu, \frac{\sigma^2}{n}\right]$$

If $\sigma^2/n$ is the variance of the sampling distribution, then the standard deviation is $\sigma/\sqrt{n}$. This is commonly referred to as the ***standard error of the mean***.

# Confidence Interval on a Mean

## (*n* large)




An equation for the (1–α) × 100% confidence interval on a mean:

$$\bar{x} \pm z_{\alpha/2}\left(\frac{\sigma}{\sqrt{n}}\right)$$

where $z_{\alpha/2}$ is the critical point corresponding to a tail area of $\alpha/2$

This equation can be used as long as $n \geq 30$, even if $\sigma$ is unknown.

# Example

A confidence interval is desired for the true average stray load loss μ (watts) for a certain type of induction motor when the line current is held at 10 amps for a speed of1500 rpm.  Assume that σ = 3.0 for stray load loss.

a. Compute a 95% confidence interval for μ when n = 25 and $\bar{x} =$ 56.8.
b. Compute a 99% confidence interval for μ when n = 75 and $\bar{x} =$ 56.8.
b. How large must n be if the length of the 99% confidence interval is to be 1.0?

# What if *n* is small? Student's t Distribution

- As the sample size becomes smaller, the sample standard deviation becomes an increasingly poor approximation of the population standard deviation. The end result is that a 95% confidence interval computed using s instead of σ may actually only contain the population mean 90% of the time, or 85% of the time, or even less.

- William Gosset developed a new probability distribution, which he called the t distribution, to describe the probabilities associated with the statistic

$$t = \frac{\bar{x} - \mu}{s / \sqrt{n}}$$

# What if *n* is small? Student's t Distribution

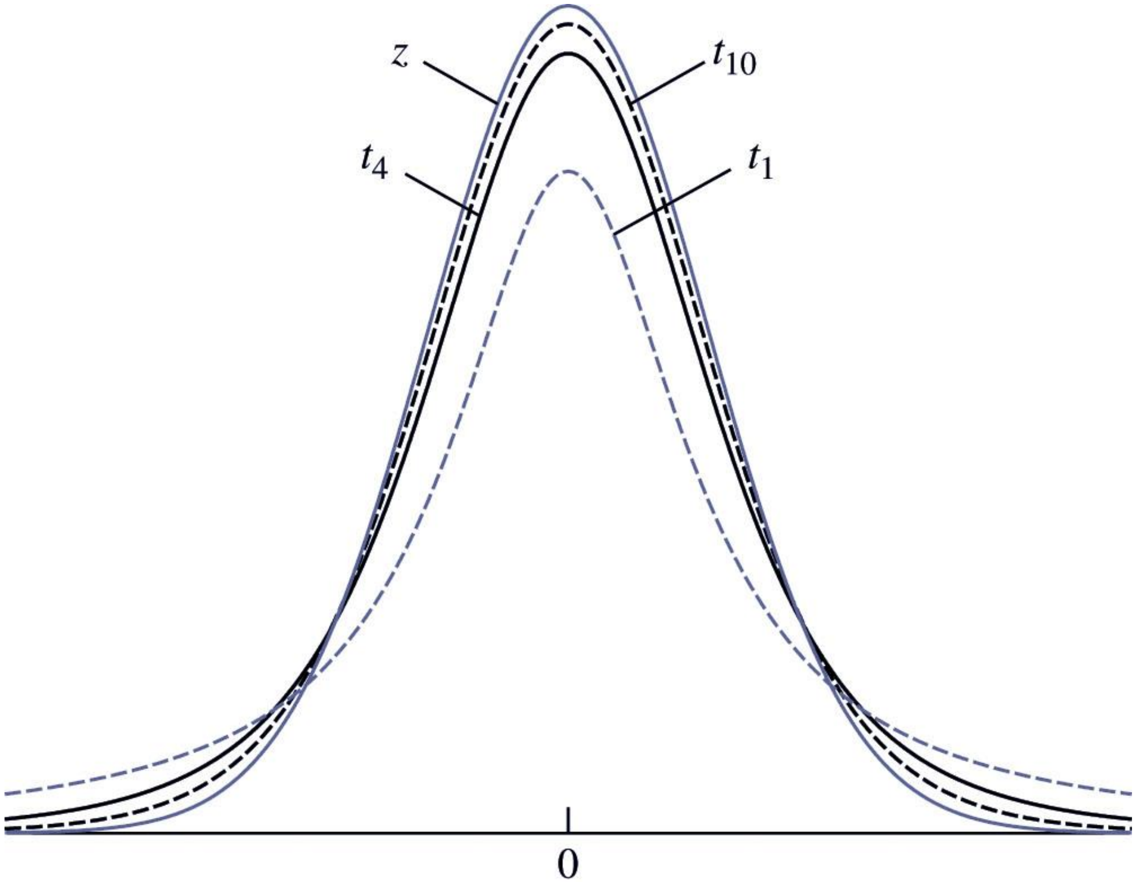



Figure 5.9

# Confidence Interval on a Mean

## (σ UNKNOWN, n small)

An equation for the $(1-\alpha)\times 100\%$ confidence interval on a mean:

$$\bar{x} \pm t_{\alpha/2,n-1}\left(\frac{s}{\sqrt{n}}\right)$$

Where $t_{\alpha/2,n-1}$ is the critical point corresponding to a tail area of $\alpha/2$.

# Student's t Distribution

Upper critical values of Student's t distribution with v degrees of freedom

Probability of exceeding the critical value

| v | 0.10 | 0.05 | 0.025 | 0.01 | 0.005 | 0.001 |
|---|---|---|---|---|---|---|
| 1. | 3.078 | 6.314 | 12.706 | 31.821 | 63.657 | 318.313 |
| 2. | 1.886 | 2.920 | 4.303 | 6.965 | 9.925 | 22.327 |
| 3. | 1.638 | 2.353 | 3.182 | 4.541 | 5.841 | 10.215 |
| 4. | 1.533 | 2.132 | 2.776 | 3.747 | 4.604 | 7.173 |
| 5. | 1.476 | 2.015 | 2.571 | 3.365 | 4.032 | 5.893 |
| 6. | 1.440 | 1.943 | 2.447 | 3.143 | 3.707 | 5.208 |
| 7. | 1.415 | 1.895 | 2.365 | 2.998 | 3.499 | 4.782 |
| 8. | 1.397 | 1.860 | 2.306 | 2.896 | 3.355 | 4.499 |
| 9. | 1.383 | 1.833 | 2.262 | 2.821 | 3.250 | 4.296 |
| 10. | 1.372 | 1.812 | 2.228 | 2.764 | 3.169 | 4.143 |

# Example

The results of a Wagner turbidity test performed on 15 samples of standard Ottawa testing sand were (in microamperes):

| Sample means |
|---|
| 26.7 |
| 25.8 |
| 24.0 |
| 24.9 |
| 26.4 |
| 25.9 |
| 24.4 |
| 21.7 |
| 24.1 |
| 25.9 |
| 27.3 |
| 26.9 |
| 27.3 |
| 24.8 |
| 23.6 |

Find a 95% confidence interval for $\mu$, the true average Wagner turbidity of all such sand samples.

# Confidence Interval on Differences ($\sigma_1$ and $\sigma_2$ KNOWN)

An equation for the $(1-\alpha)\times 100\%$ confidence interval on a difference in means:

$$(\bar{x}_1 - \bar{x}_2) \pm z_{\alpha/2}\sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

where $z_{\alpha/2}$ is the critical point corresponding to a tail area of $\alpha/2$

This relationship is exact if the two populations are normally distributed. Otherwise, the confidence interval is approximately valid for large sample sizes ($n_1 \geq 30$ and $n_2 \geq 30$).

# Example

Tensile tests were performed on two different grades of aluminum spars used in manufacturing the wing of a commercial transport aircraft. From past experience with the spar manufacturing process and the testing procedure, the standard deviations of tensile strengths are assumed to be known. The data are shown below.

| Spar Grade | Sample Size | Sample Mean Tensile Strength ($kg/mm^2$) | Standard Deviation ($kg/mm^2$) |
|---|---|---|---|
| 1 | 30 | 92.5 | 1.0 |
| 2 | 40 | 78.5 | 1.5 |

Find a 90% confidence interval on the difference in mean strength.

# Confidence Interval on Differences
# ($\sigma_1$ and $\sigma_2$ UNKNOWN but equal)

If random samples of size $n_1$ and $n_2$ are drawn from two *normal populations* with equal but unknown variances, a $100(1-\alpha)\%$ confidence interval on the difference between the sample means, $\mu_1 - \mu_2$ is:

$$(\bar{x}_1 - \bar{x}_2) \pm t_{\alpha/2, n_1+n_2-2} S_p \sqrt{\frac{1}{n_1} + \frac{1}{n_2}}$$

where $S_p$ is a "pooled" estimator of the unknown standard deviation and is calculated as:

$$S_p = \sqrt{\frac{(n_1 - 1)s_1{}^2 + (n_2 - 1)s_2{}^2}{n_1 + n_2 - 2}}$$

But this can only be used if both populations are ***normally distributed***.

# Example

A machine is used to fill plastic bottles with bleach. A sample of 18 bottles had a mean fill volume of 2.007 L and a standard deviation of 0.010 L. The machine was then moved to another location. A sample of 10 bottles filled at the new location had a mean fill volume of 2.001 L and a standard deviation of 0.012 L. It is believed that moving the machine may have changed the mean fill volume, but is unlikely to have changed the standard deviation. Assume that both samples come from approximately normal populations. Find a 99% confidence interval for the difference between the mean fill volumes at the two locations.

# Confidence Interval on Differences ($\sigma_1$ and $\sigma_2$ UNKNOWN and unequal)

If $\bar{x}_1$, $\bar{x}_2$, $s_1^2$, and $s_2^2$ are the means and variances of two random samples of sizes $n_1$ and $n_2$, respectively, from two independent normal populations with unknown and unequal variances, an approximate $100(1-\alpha)\%$ confidence interval on the difference in means $\mu_1 - \mu_2$ is

$$\bar{x}_1 - \bar{x}_2 - t_{\alpha/2,v}\sqrt{\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}} \le \mu_1 - \mu_2 \le \bar{x}_1 - \bar{x}_2 + t_{\alpha/2,v}\sqrt{\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}} \qquad (10\text{-}20)$$

where $v$ is given by Equation 10-16 and $t_{\alpha/2,v}$ is the upper $\alpha/2$ percentage point of the $t$ distribution with $v$ degrees of freedom.

$$v = \frac{\left(\dfrac{S_1^2}{n_1}+\dfrac{S_2^2}{n_2}\right)^2}{\dfrac{(S_1^2/n_1)^2}{n_1-1}+\dfrac{(S_2^2/n_2)^2}{n_2-1}} \qquad (10\text{-}16)$$

# Confidence Intervals on Paired Samples

An equation for the $(1-\alpha)\times 100\%$ confidence interval on $\bar{d}$ for a paired sample:

$$\bar{d} \pm t_{\alpha/2,n-1}\left(\frac{s_d}{\sqrt{n}}\right)$$

But this can only be used if both populations are ***normally distributed***.

# Example

The table below shows the time for 14 subjects to parallel park two cars. Find a 90% confidence interval for $\mu_D$. What can we conclude?

| *Subject* | *Automobile 1* | *Automobile 2* | *Difference* |
|---|---|---|---|
| 1 | 37.0 | 17.8 | 19.2 |
| 2 | 25.8 | 20.2 | 5.6 |
| 3 | 16.2 | 16.8 | -0.6 |
| 4 | 24.2 | 41.4 | -17.2 |
| 5 | 22.0 | 21.4 | 0.6 |
| 6 | 33.4 | 38.4 | -5.0 |
| 7 | 23.8 | 16.8 | 7.0 |
| 8 | 58.2 | 32.2 | 26.0 |
| 9 | 33.6 | 27.8 | 5.8 |
| 10 | 24.4 | 23.2 | 1.2 |
| 11 | 23.4 | 29.6 | -6.2 |
| 12 | 21.2 | 20.6 | 0.6 |
| 13 | 36.2 | 32.2 | 4.0 |
| 14 | 29.8 | 53.8 | -24.0 |

# Confidence Interval on the Variance

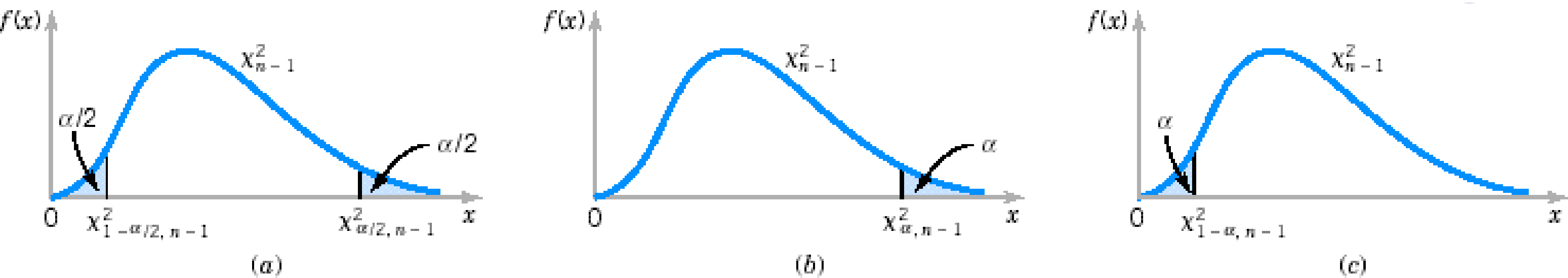
If a random sample of size n is taken from a normally distributed population, a 100(1–α)% confidence interval on the variance of the population is:

$$\frac{(n-1)s^2}{\chi^2_{\alpha/2,n-1}} \le \sigma^2 \le \frac{(n-1)s^2}{\chi^2_{1-\alpha/2,n-1}}$$

But this can only be used if the population is ***normally distributed***.

Here, $\chi^2_{\alpha/2,n-1}$ and $\chi^2_{1-\alpha/2,n-1}$ are the upper and lower critical points of the chi-square distribution with $n$-1 degrees of freedom. Because the $\chi^2$ distribution is asymmetrical, the upper and lower tails are *not* the same.

# Chi-Square Distribution



# Example

A manufacturer of soft drink beverages is interested in the uniformity of the machine used to fill cans. Specifically, it is desirable that the standard deviation, $\sigma$, of the filling process be less than 0.2 fluid ounces; otherwise there will be a higher than allowable percentage of cans that are underfilled. We will assume that fill volume is approximately normally distributed. A random sample of 20 cans result in a sample variance of 0.0225 (fluid ounces)$^2$. Find a 95% upper-confidence interval on the variance and the standard deviation.

# Confidence Interval on Ratio of Variances ($\sigma_1$ and $\sigma_2$ UNKNOWN):

A 100(1–α)% confidence interval on the ratio of variances (assuming both populations are normally distributed) is:

$$\frac{s_1^2}{s_2^2}\frac{1}{F_{\alpha/2,v_1,v_2}} \le \frac{\sigma_1^2}{\sigma_2^2} \le \frac{s_1^2}{s_2^2} F_{\alpha/2,v_2,v_1}$$

$$where:$$

$$v_1 = n_1 - 1; v_2 = n_2 - 1$$

$$F_{1-\alpha,u,v} = \frac{1}{F_{\alpha,v,u}}$$

# F Distribution

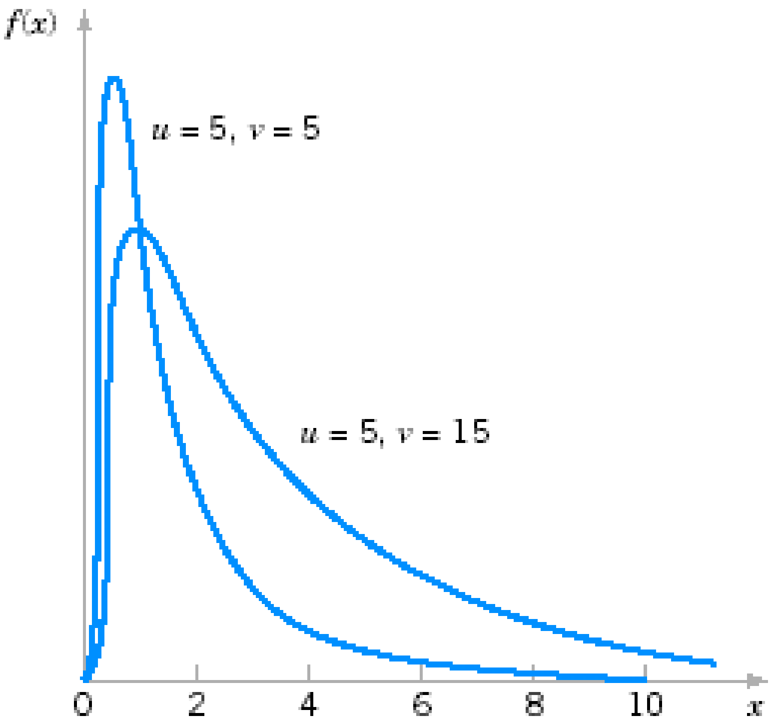



**Figure 10-4** Probability density functions of two $F$ distributions.

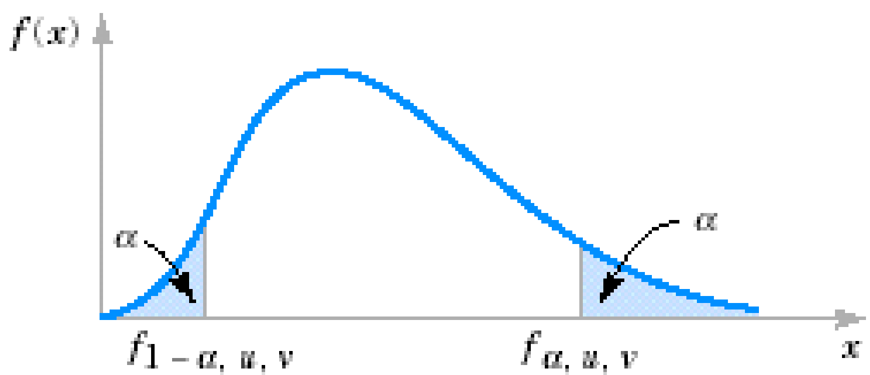



**Figure 10-5** Upper and lower percentage points of the $F$ distribution.

# Example

In a batch chemical process used for etching printed circuit boards, two different catalysts are being compared to determine whether they require different emersion times for removal of identical quantities of photoresist material. Twelve batches were run with catalyst 1, resulting in a sample mean emersion time of 24.6 minutes and a sample standard deviation of 0.85 minutes. Fifteen batches were run with catalyst 2, resulting in a mean emersion time of 22.1 minutes and a standard deviation of 0.98 minutes. Find a 90% confidence interval on the ratio of variances.

# Prediction Interval

- In some problem situations, we may be interested in predicting a future observation of a variable.
- This is a different problem than estimating the mean of that variable, so a confidence interval is not appropriate.

**Prediction Interval**

**A 100(1 - α)% prediction interval (PI) on a single future observation from a normal distribution** is given by

$$\bar{x} - t_{\alpha/2,n-1}s\sqrt{1+\frac{1}{n}} \le X_{n+1} \le \bar{x} + t_{\alpha/2,n-1}s\sqrt{1+\frac{1}{n}} \qquad (8\text{-}27)$$

# Example

The comprehensive strength of concrete is being tested by a civil engineer who tests 12 specimens and obtains the following data

| | | | |
|---|---|---|---|
| 2216 | 2237 | 2249 | 2204 |
| 2225 | 2301 | 2281 | 2263 |
| 2318 | 2255 | 2275 | 2295 |

Compute a 90% prediction interval on the next specimen of concrete tested.